

# Weil Representation of a Generalized Linear Group over a Ring of Truncated Polynomials over a Finite Field Endowed with a Second Class Involution

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**Abstract.** We construct a complex linear Weil representation  $\rho$  of the generalized special linear group  $G = \mathrm{SL}_*^1(2, A_n)$  ( $A_n = K[x]/\langle x^n \rangle$ ,  $K$  the quadratic extension of the finite field  $k$  of  $q$  elements,  $q$  odd), where  $A_n$  is endowed with a second class involution. After the construction of a specific data, the representation is defined on the generators of a Bruhat presentation of  $G$ , via linear operators satisfying the relations of the presentation. The structure of a unitary group  $U$  associated to  $G$  is described. Using this group we obtain a first decomposition of  $\rho$ .

*Key words:* Weil representation; generalized special linear group

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## 1 Introduction

Weil representations is a central topic in representation theory. They arise as a consequence of one of the many seminal works of A. Weil [17]. They are projective representations of the groups  $\mathrm{Sp}(2n, F)$ ,  $F$  a locally compact field. These representations are an important subject of study. By decomposition into irreducible factors, Weil representations provide all irreducible complex representations of the general linear group over a finite field, and over a local field of residual characteristic different from 2. It has many other important consequences, and it has applications in different topics as theta functions and physics to mention only some of them. In fact, in representation theory it helps to understand (among other things) the harmonic analysis of the group, and in the Langlands program, to explain the relations between linear groups defined over local or global fields, and the Galois groups of the fields.

A point of view that has been favorably used in representation theory, is to extend to higher rank groups, methods successfully used in lower rank groups. This philosophy has led Pantoja and Soto-Andrade (see [11, 12, 13]) to define the groups  $\mathrm{GL}_*^\varepsilon(2, A)$  and  $\mathrm{SL}_*^\varepsilon(2, A)$ , where  $\varepsilon = \pm 1$ . These are “generalized linear groups” of rank 2 with coefficients in a unitary involutive ring  $(A, *)$  [10] ( $\varepsilon = \pm 1$ ). In this way, the symplectic group and the orthogonal group of rank  $n$  over a field  $F$  appear as groups  $\mathrm{SL}_*^\varepsilon(2, A)$  considering as involutive ring, the ring of  $n \times n$  matrices over  $F$  with the transposition of matrices, and taking, respectively,  $\varepsilon$  equals to  $-1$  and  $1$ . Different choices of involutive rings produce new examples of diverse kind.

The rings considered are, in general, non-commutative, and this non-commutativity is controlled by the relation  $(ab)^* = b^*a^*$ . In this sense, it should be pointed out the similarity of our

group  $\mathrm{GL}_*(2, A)$ , with the quantum group  $\mathrm{GL}^q(2)$  (as a “group” of matrices with some commuting relations). Furthermore, the generalized linear groups afford the notion of a  $*$ -determinant,  $\det_*$ , in analogy with the  $q$ -determinant. Also, the homomorphism  $*$ -determinant can be considered as a first way of describing the mentioned groups, as these can be defined as two by two matrices with coefficients in  $A$ , (satisfying some commutation relations that involve  $*$ ), with  $*$ -determinant different from 0. Then, the groups  $\mathrm{SL}_*(2, A)$  appear as the kernel of the homomorphism  $\det_*$ .

The approach, used here, has been considered in diverse interesting groups [6, 7, 16]. Having a presentation of the group, that the authors call Bruhat presentation, and certain specific data, a generalized Weil representation has been constructed. Historically, the groups  $G$  were defined first for  $\varepsilon = -1$ . The work of Soto-Andrade on the symplectic group over a finite field [15] and the representations constructed by Gutiérrez [6] appear as examples of these constructions in [7]. Later, after the generalization to  $\varepsilon = 1$ , Vera [16] constructed a generalized Weil representation of the orthogonal group over a finite field, where she also relates the representation with the one that can be constructed by the theory of dual pairs of Howe [9]. However, even that the orthogonal group is a natural example of a generalized linear group with  $\varepsilon = 1$ , her work is done treating the group as a “symplectic type” group, i.e., as a  $\mathrm{SL}_*^{-1}$  group, by twisting the transposition of matrices to define the involution of the ring.

Our method to construct Weil representations is one of many successful approaches to attack this topic, and has been studied by several authors (see, e.g., [1, 4, 5, 17]). Using Weil’s original point of view, Szechtman et al. in [2] construct Weil representations of symplectic groups over finite rings via Heisenberg groups. Recently, Herman and Szechtman [8] construct Weil representations of unitary groups associated to a finite, commutative, local principal ring of odd characteristic, by imbedding the group into a symplectic group.

A great diversity of groups can be originated via generalized linear groups for appropriate choices of involutive rings. For each of them, Weil representations could be constructed by using our approach of defining linear operators for each generator of the group in such a way that they satisfy the basic (universal) relations of a “simple” presentation. This variety of cases for which a (generalized) Weil representation could be produced has led us to verify that, in practice, the procedure is effective. In fact, several examples of groups  $\mathrm{GL}_*^\varepsilon(2, A)$  and  $\mathrm{SL}_*^\varepsilon(2, A)$  have been given in loc. cit. for different choices of involutive rings provided with first class involutions [10]. However, no example with an involutive ring (algebra) provided with a second class involution has been considered up to now. Furthermore, explicit constructions for generalized linear groups with  $\varepsilon = 1$  have not been made so far.

In this work, we construct a Weil representation of a generalized  $\mathrm{SL}_*^1$  “orthogonal type group” (i.e., a generalized special linear group with  $\varepsilon = 1$ ) over the ring  $A_n = K[x]/\langle x^n \rangle$ , which is a non-semisimple ring (algebra in fact) over  $K$ ,  $K$  the quadratic extension of the finite field  $k$  of  $q$  elements ( $q$  odd), provided with a second class involution. We show first that the group under consideration has a Bruhat presentation, after which a data necessary to produce a Weil representation of  $G$  via relations and generators, is specifically described. We also obtain a first decomposition of the representation. Using a complete different approach leaning on the works of Amritanshu Prasad and Kunal Dutta [3, 14], a further decomposition could be achieved. Furthermore, a comparison of their methods and ours will be performed in a work that will appear elsewhere.

The paper is organized as follows: In Section 2, we present the main definitions on generalized classical groups  $\mathrm{GL}_*^\varepsilon(2, A)$  and  $\mathrm{SL}_*^\varepsilon(2, A)$  for an involutive ring  $(A, *)$  and we describe some properties of the truncated polynomials ring  $K[x]/\langle x^n \rangle$ , for  $K$  a quadratic extension of a field of  $q$  elements ( $q$  odd). In Section 3, a Bruhat presentation of  $\mathrm{SL}_*^1(2, A_n)$  is constructed. Section 4 is devoted first to recall a very general procedure to construct generalized Weil representations of  $\mathrm{SL}_*^\varepsilon(2, A)$ , for a group with a Bruhat presentation and a suitable data  $(M, \chi, \gamma, c)$ . After this, the necessary data for the group under consideration is produced and completely described and

detailed. Finally, in Section 5, we define the abelian “unitary” group  $U(M, \chi, \gamma, c)$  of  $(M, \chi, \gamma, c)$ , with the explicit decomposition into cyclic subgroups, which allow us to get a first decomposition of the constructed representation.

## 2 Preliminaries

In this section, we fix notations and recall some basic facts about generalized general special linear groups over involutive rings.

Let  $A$  be a unitary ring endowed with an involution  $*$ , i.e., an antiautomorphism  $a \mapsto a^*$  of  $A$  of order two. We denote by  $Z(A)$  the center of  $A$ , and we write  $A^\times$  (respectively  $Z(A)^\times$ ) for the group of invertible elements of  $A$  (respectively, of  $Z(A)$ ).  $T^{\text{sym}}$  (respectively  $T^{\text{asym}}$ ) stands for the set of symmetric (respectively antisymmetric) elements of the subset  $T$  of  $A$ , i.e., the set of elements  $a$  in  $T$  such that  $a^* = a$  (respectively  $a^* = -a$ ). The involution  $*$  induces an involution on the ring of  $2 \times 2$  matrices with coefficients in  $A$ , by  $(g^*)_{ij} = g_{ji}^*$  ( $g \in M(2, A)$ ), which we denote also with the symbol  $*$ .

### 2.1 The groups $\text{SL}_*^1(2, A)$

We give a brief description of the groups  $\text{SL}_*^1(2, A)$ . For more details see [13].

If  $A$  is a unitary ring with an involution  $*$ , and  $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  in  $M(2, A)$ , let  $\text{GL}_*^1(2, A) := \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2, A) : g^* J g = \lambda(g) J, \lambda(g) \in (Z(A)^{\text{sym}})^\times \right\}$ . Then  $\text{GL}_*^1(2, A)$  is a group. Moreover, the map

$$\det_* : \text{GL}_*^1(2, A) \rightarrow (Z(A)^{\text{sym}})^\times$$

given by  $\det_*(g) = \lambda(g) = ad^* + bc^* = a^*d + c^*b$  is an homomorphism.

We define  $\text{SL}_*^1(2, A)$  as the kernel of  $\det_*$ . One can observe that the entries of  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_*^1(2, A)$  satisfy:  $a^*c = -c^*a$ ,  $ab^* = -ba^*$ ,  $b^*d = -d^*b$ ,  $cd^* = -dc^*$ .

### 2.2 The involutive ring of truncated polynomials

Let  $k$  be a finite field of  $q$  elements, where  $q$  is a power of an odd prime  $p$ . We consider  $K$  the unique quadratic extension of  $k$  and we take  $\Delta \in K$  such that  $K = k(\Delta)$  and  $\Delta^2 \in k$ . We write  $\overline{a + b\Delta}$  to denote the image  $a - b\Delta$  of the element  $a + b\Delta$  under the nontrivial element of the Galois group of the extension  $K/k$ . Let

$$A_n = K[x]/\langle x^n \rangle, \quad n \in \mathbb{N},$$

which will be considered as polynomials (with coefficients in  $K$ ), truncated at  $n$  (i.e., such that  $x^m = 0$  for  $m \geq n$ ).

We define an involution  $*$  in the  $k$ -algebra  $A_n$  by

$$\begin{aligned} a + b\Delta &\mapsto \overline{a + b\Delta}, \\ x &\mapsto -x. \end{aligned}$$

We first present some results concerning cardinalities about sets that we will use later on.

**Proposition 1.** *If  $|S|$  denotes the cardinality of  $S$ , we have*

1. *The order of the group of invertible elements of  $A_n$  is  $|A_n^\times| = (q^2 - 1)q^{2(n-1)}$ .*

2.

$$A_n^{\text{sym}} = \left\{ a = \sum_{i=0}^{n-1} a_i x^i : a_{2i} \in k \text{ and } a_{2i+1} \in \Delta k \right\}$$

has cardinality  $|A_n^{\text{sym}}| = q^n$ .

3.

$$A_n^{\text{asym}} = \left\{ a = \sum_{i=0}^{n-1} a_i x^i : a_{2i} \in \Delta k \text{ and } a_{2i+1} \in k \right\}$$

has cardinality  $|A_n^{\text{asym}}| = q^n$ .

**Proof.** 1. The result is clear observing that an invertible element of the ring must have nonzero constant term.

2. Let  $a = \sum_{i=0}^{n-1} a_i x^i$  be an arbitrary element in  $A_n$ , then

$$a^* = \sum_{i=0}^{n-1} (-1)^i \overline{a_i} x^i.$$

So  $a^* = a$  if and only if  $a_i = (-1)^i \overline{a_i}$ , for each  $i$ . Thus

$$A_n^{\text{sym}} = \left\{ a = \sum_{i=0}^{n-1} a_i x^i : a_i \in k \text{ for } i \text{ even and } a_i \in \Delta k \text{ for } i \text{ odd} \right\},$$

and the result follows.

3. Similar to 2. This completes the proof. ■

**Proposition 2.** *The order of  $\text{SL}_*^1(2, A_n)$  is  $(q^2 - 1)q^{4n-3}(q + 1)$ .*

**Proof.** The group  $\text{SL}_*(2, A_n)$  acts on  $M_{2 \times 1}(A_n)$  by left multiplication. Set

$$O_1 = \left\{ \begin{pmatrix} a \\ ua \end{pmatrix} \in M_{2 \times 1}(A_n) : a \in A_n^\times, u \in A_n^{\text{asym}} \right\},$$

$$O_2 = \left\{ \begin{pmatrix} uc \\ c \end{pmatrix} \in M_{2 \times 1}(A_n) : c \in A_n^\times, u \in A_n^{\text{asym}} \setminus A_n^\times \right\}.$$

We claim that the orbit  $\text{Orb}_{\text{SL}_*^1(2, A_n)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is the union of  $O_1$  and  $O_2$ . In fact, we note that the first column  $\begin{pmatrix} a \\ c \end{pmatrix}$  of a matrix in  $\text{SL}_*^1(2, A_n)$  satisfies  $a^*c = -c^*a$ . Since  $a$  or  $c$  must be invertible, we get  $ca^{-1}$  (or  $ac^{-1}$ ) is anti-symmetric, then  $c = ua$  (or  $a = uc$ ), for some anti-symmetric element  $u$ . From here  $\text{Orb}_{\text{SL}_*^1(2, A_n)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is contained in the union  $O_1 \cup O_2$ . On the other hand, if  $\begin{pmatrix} a \\ ua \end{pmatrix} \in O_1$ , then

$$\begin{pmatrix} a & 0 \\ ua & a^{*-1} \end{pmatrix} \in \text{SL}_*^1(2, A_n), \quad \begin{pmatrix} a \\ ua \end{pmatrix} = \begin{pmatrix} a & 0 \\ ua & a^{*-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Thus,  $O_1$  is contained in the orbit  $\text{Orb}_{\text{SL}_*^1(2, A_n)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Similarly,  $\text{Orb}_{\text{SL}_*^1(2, A_n)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  contains  $O_2$ . This proves the claim.

Now given that the sets are disjoint we verify

$$\left| \text{Orb}_{\text{SL}_*^1(2, A_n)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| = (q^2 - 1)q^{2(n-1)}q^n + (q^2 - 1)q^{2(n-1)}q^{n-1} = (q^2 - 1)q^{3n-3}(q + 1).$$

Finally, the isotropy group  $\text{Stab} \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$  is the group of upper unipotent matrices in  $\text{SL}_*^1(2, A_n)$  which has cardinality  $q^n$ , and therefore our proposition follows.  $\blacksquare$

### 3 Bruhat presentation for $\text{SL}_*^1(2, A_n)$

In this section, we prove that the group  $G = \text{SL}_*^1(2, A_n)$  has a Bruhat-like presentation, which will be used in the construction of a Weil representation of  $G$ .

To this end, we set

$$h_t = h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{*-1} \end{pmatrix}, \quad t \in A^\times, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$u_s = u(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad s \in A^{\text{asym}},$$

noticing that the matrices  $h_t, \omega, u_s \in \text{SL}_*^1(2, A)$ .

**Lemma 1.** *Let  $a$  and  $c$  be two elements in  $A_n$  such that  $Aa + Ac = A$  and  $a^*c = c^*a$ . Then, there exists an element  $s \in A_n^{\text{asym}}$  such that  $a + sc$  is invertible.*

**Proof.** We first observe that  $a$  or  $c$  has to be invertible. Suppose that  $a$  is invertible, then considering  $s = 0$  we have the result. On the other hand, if  $a$  is non-invertible, then  $c$  is invertible and so any nonzero element  $s$  in  $k\Delta$  proves the lemma.  $\blacksquare$

**Proposition 3.** *The matrices  $h_t, u_s$  and  $w$ , with  $t \in A_n^\times, s \in A_n^{\text{asym}}$ , generate  $\text{SL}_*^1(2, A_n)$ .*

**Proof.** Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_*^1(2, A_n)$ . If  $c = 0$ , then  $g = h_a u_{a^{-1}b}$ .

Suppose now that  $c$  is invertible. Since  $\det_*(g) = 1$ , i.e.,  $ad^* + bc^* = 1$  one gets  $b = c^{*-1} + ac^{-1}d$ . One checks  $g = h(c^{*-1})u(c^*a)wu(c^{-1}d)$ .

Now if  $c \neq 0$  and non-invertible, we take an antisymmetric element  $s$  satisfying the above lemma. Now

$$wu_s g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a + sc & b + sd \end{pmatrix}.$$

Thus the matrix  $wu_s g$  has entry  $(2, 1)$  invertible and we can write

$$wu(s)g = h((a + sc)^{*-1})u((a + sc)^*c)wu((a + sc)^{-1}(b + sd)).$$

From here,  $\text{SL}_*^1(2, A_n)$  is generated by the matrices  $h_t, u_s$  and  $w$  with  $t \in A_n^\times, s \in A_n^{\text{asym}}$ .  $\blacksquare$

**Lemma 2.** *Let  $a$  and  $b$  be two non-invertible antisymmetric elements in  $A_n$ . Then, we can find an antisymmetric invertible element  $v \in A_n$  such that  $a - v^{-1}$  and  $b + v$  are antisymmetric invertible elements in  $A_n$ .*

**Proof.** Since  $a, b$  are non-invertible elements, they are in the ideal generated by  $x$ . Then taking any nonzero element  $v \in \Delta k$ , we check that  $a - v^{-1}$  and  $b + v$  are antisymmetric invertible elements in  $A_n$ . ■

Lemmas 1, 2 and Proposition 3 prove, using the same argument as in Theorem 15 of [11], our next result:

**Theorem 1.** *The matrices  $h_t, u_s$  and  $w$  ( $t \in A_n^\times, s \in A_n^{\text{asym}}$ ), with the commuting relations:*

$$\begin{aligned} h_{t_1} h_{t_2} &= h_{t_1 t_2}, \\ u_{s_1} u_{s_2} &= u_{s_1 + s_2}, \\ h_t u_s &= u_{t s t^*} h_t, \\ w^2 &= 1, \\ h_t w &= w h_{t^{-1}}, \\ w u_{t^{-1}} w u_{-t} w u_{t^{-1}} &= h_{-t} \end{aligned}$$

give a presentation of  $\text{SL}_*^1(2, A_n)$ .

## 4 A generalized Weil representation of $\text{SL}_*^1(2, A_n)$

In [7, Theorem 4.4], a Weil representation is constructed for groups that afford a Bruhat presentation as the one obtained in Theorem 1. Specific data are required.

First, we must have a finite right  $A$ -module  $M$  and pair of functions  $\chi: M \times M \rightarrow \mathbb{C}^\times$  and  $\gamma: A^{\text{asym}} \times M \rightarrow \mathbb{C}^\times$ , a nonzero complex number  $c$ , and a character  $\alpha \in \widehat{A^\times}$ , satisfying the following properties and relations among them:

- a)  $\chi$  is bi-additive;
- b)  $\chi(mt, v) = \alpha(tt^*)\chi(m, vt^*)$  for  $m, v \in M$  and  $t \in A^\times$ ;
- c)  $\chi(v, m) = [\chi(m, v)]^{-1}$ ;
- d)  $\chi(v, m) = 1$  for all  $m \in M$ , implies  $v = 0$ ;
- e)  $\gamma(b, mt) = \gamma(tbt^*, m)$ ;
- f)  $\gamma(t, m + z) = \gamma(t, m)\gamma(t, z)\chi(m, zt)$  for all  $m, z \in M, t \in A^{\text{asym}}$ , where  $t$  is anti-symmetric invertible in  $A$  and  $c \in \mathbb{C}^\times$  satisfies  $c^2 |M| = 1$ ;
- g)  $\gamma(b + b', m) = \gamma(b, m)\gamma(b', m)$ , for all  $b, b' \in A^{\text{asym}}$  and  $m \in M$ , and
- h)  $\sum_{m \in M} \gamma(t, m) = \frac{\alpha(t)}{c}$ .

Then, with the above data we have (see [7]):

**Theorem 2.** *If  $\text{SL}_*^1(2, A)$  has a Bruhat presentation, the data  $(M, \chi, \gamma, c)$  defines a (linear) representation  $(\mathbb{C}^M, \rho)$  of  $\text{SL}_*^1(2, A)$ , which we call Weil representation, by*

- 1)  $\rho_{u_b}(e_a) = \gamma(b, a)e_a$ ,
- 2)  $\rho_{h_t}(e_a) = \alpha(t)e_{at^{-1}}$ ,
- 3)  $\rho_w(e_a) = c \sum_{b \in M} \chi(-a, b)e_b$ ,

for  $a \in M, b \in A^{\text{asym}}, t \in A^\times$  and  $e_a$  the Dirac function at  $a$ , defined by  $e_a(u) = 1$  if  $u = a$  and  $e_a(u) = 0$  otherwise.

#### 4.1 Construction of data for $\mathrm{SL}_*^1(2, A_n)$

Since we already know that  $\mathrm{SL}_*^1(2, A_n)$  has a Bruhat presentation, we construct now a data for this group, in order to apply Theorem 2.

Let  $\psi_0$  be a nontrivial additive character of  $K$  such that  $\psi_0$  is nontrivial in  $k$  and in  $\Delta k$ . We consider the biadditive function  $\chi$  from  $A_n \times A_n$  to  $\mathbb{C}^\times$  given by

$$(a, b) \mapsto \psi(a^*b),$$

where  $\psi$  is the nontrivial character of  $A_n$  defined as

$$\psi(a_o + a_1x + \cdots + a_{n-1}x^{n-1}) = \psi_0(a_{n-1}).$$

We take  $M = A_n$  and we assume  $\alpha = \mathbf{1}$ . It is clear that a), b) and c) above are fulfilled. We prove now d).

**Proposition 4.** *The biadditive map  $\chi$  is non-degenerate.*

**Proof.** Let  $a$  be a nonzero element of  $A_n$ . We need to prove that there is an element  $b$  in  $A_n$  such that  $\chi(a, b) \neq 1$ . Let us write  $a = a_o + a_1x + \cdots + a_{n-1}x^{n-1}$ . If  $a_i$  is the first nonzero coefficient of  $a$ , set  $b = tx^{n-i-1}$ . Then  $\chi(a, b) = \psi_0((-1)^i \bar{a}_i t)$ . If  $t$  runs over  $K$ , then so does  $(-1)^i \bar{a}_i t$ . Since the character  $\psi_0$  is nontrivial, the result follows. ■

We now define the function  $\gamma$  by  $\gamma(t, m) = \chi(-2^{-1}tm, m)$ , for  $t \in A_n^{\mathrm{asym}}$  and  $m$  in  $A_n$ . We claim that  $\gamma$  satisfies conditions e), f), g), and h) above.

e) Let  $b \in A_n^{\mathrm{asym}}$ ,  $m \in A_n$  and  $t \in A_n^\times$ . Then

$$\gamma(tbt^*, mt^{-1}) = \chi(-2^{-1}tbt^*mt^{-1}, mt^{-1}) = \chi(-2^{-1}bt^*m, mt^{-1}).$$

Now, using that  $\chi$  is balanced, we get  $\gamma(tbt^*, mt^{-1}) = \chi(-2^{-1}bm, m) = \gamma(b, m)$ .

f) Let  $t \in A_n^{\mathrm{asym}}$  and  $m, v \in A_n$ . Then

$$\begin{aligned} \gamma(t, m+v) &= \chi(-2^{-1}t(m+v), m+v) \\ &= \chi(-2^{-1}tm, m)\chi(-2^{-1}tv, v)\chi(-2^{-1}tm, v)\chi(-2^{-1}tv, m). \end{aligned}$$

Now, since  $\chi$  is balanced, we have  $\chi(-2^{-1}tm, v) = \chi(m, 2^{-1}tv)$ . On the other hand, we have  $\chi(-2^{-1}tv, m) = \chi(m, 2^{-1}tv)$ . Then  $\chi(-2^{-1}tm, v)\chi(-2^{-1}tv, m) = \chi(m, vt)$ , and

$$\gamma(t, m+v) = \chi(-2^{-1}tm, m)\chi(-2^{-1}tv, v)\chi(m, vt).$$

g)  $\gamma(b_1 + b_2, m) = \gamma(b_1, m)\gamma(b_2, m)$  follows from the bi-additive property of  $\chi$ .

h) Set  $c = (-1)^n \frac{1}{q^n}$ . Since  $|A_n| = q^{2n}$ , we have  $c^2|A_n| = 1$ . The result will follow from Proposition 5 below, which needs Lemma 3:

**Lemma 3.** *Let  $\psi$  be the additive character of  $K$  defined above. Then*

$$\sum_{z \in K} \psi(N_{K/k}(\lambda z)) = \begin{cases} q^2 & \text{if } \lambda = 0, \\ -q & \text{if } \lambda \neq 0. \end{cases}$$

**Proof.** The case  $\lambda = 0$  is clear. Suppose then  $\lambda \neq 0$ . Since  $z$  runs over  $K$  it suffices to compute the sum  $\sum_{z \in K} \psi(N_{K/k}(z))$ . We observe

$$K = \bigcup_{t \in k} N_{K/k}^{-1}(t).$$

Since the character  $\psi$  is nontrivial on  $k$  and the cardinality of  $N_{K/k}^{-1}(t)$  is  $q + 1$  for any  $t \in k^\times$ , we have

$$\sum_{z \in K} \psi(N_{K/k}(\lambda z)) = \psi(0) + \sum_{t \in k^\times} (q + 1)\psi(t) = 1 + (q + 1)(-1) = -q. \quad \blacksquare$$

Now,

**Proposition 5.** *We have*

$$\sum_{y \in A_n} \gamma(t, y) = (-1)^n q^n.$$

**Proof.** By definition  $\gamma(t, y) = \chi(-\frac{1}{2}ty, y)$ . We set  $y = \sum_{i=0}^{n-1} \alpha_i x^i$  and  $-\frac{1}{2}t = \sum_{i=0}^{n-1} \lambda_i x^i$  for  $y \in A_n$  and  $t \in A^{\text{asym}^\times}$ . We will write sometimes  $(y)_i$  for the coefficient of  $x^i$  in  $y$ . Then

$$\sum_{y \in A_n} \gamma(t, y) = \sum_{y \in A_n} \chi\left(-\frac{1}{2}ty, y\right) = \sum_{y \in A_n} \psi\left(-\frac{1}{2}ty^*y\right) = \sum_{i=1}^n \psi_0\left(\left(-\frac{1}{2}ty^*y\right)_{n-1}\right).$$

Since  $t^* = -t$  we have  $\lambda_i = d_i \in k$  for  $i$  odd, and  $\lambda_i = d_i \Delta \in k\Delta$  for  $i$  even.

We will split the proof into two cases, according to  $n$  is even or odd. We first assume that  $n$  is even. Using the above, and after a computation, we have

$$\sum_{y \in A_n} \gamma(t, y) = \sum_{\alpha_i, i=0}^{n-1} \psi_0(d_{n-1}\beta_0 + \Delta d_{n-2}\beta_1 + \cdots + d_1\beta_{n-2} + \Delta d_0\beta_{n-1}), \quad (1)$$

where  $\beta_i = \sum_{j=0}^i (-1)^j \bar{\alpha}_j \alpha_{i-j}$ .

We first observe that  $\alpha_{n-1}$  appears only in  $\beta_{n-1}$ . Summing first over  $\alpha_{n-1}$ , we write

$$\begin{aligned} & \sum \cdots \sum_{\alpha_0, \alpha_{n-1}} \cdots \psi_0(\Delta d_0(\bar{\alpha}_0 \alpha_{n-1} - \bar{\alpha}_{n-1} \alpha_0)) \\ &= \sum \cdots \sum_{\alpha_0, \alpha_{n-1}} \cdots \psi_0(\Delta d_0(\bar{\alpha}_0 \alpha_{n-1} - \overline{\bar{\alpha}_0 \alpha_{n-1}})). \end{aligned}$$

Given the assumption on  $\psi_0$ , the character that sends  $\alpha_{n-1}$  into  $\psi_0(\Delta d_0(\bar{\alpha}_0 \alpha_{n-1} - \overline{\bar{\alpha}_0 \alpha_{n-1}}))$  is an additive, nontrivial character of  $K$ . This sum is zero unless  $\alpha_0 = 0$ . Then the sum over  $\alpha_0$  and  $\alpha_{n-1}$  contributes with  $q^2$ . In this way, the above sum becomes

$$\sum \cdots \sum_{\alpha_1, \alpha_{n-2}} \cdots q^2 \psi_0(d_{n-3}\gamma_1 + \Delta d_2\gamma_2 + \cdots + d_1\gamma_{n-3} + \Delta d_0\gamma_{n-2}),$$

where  $\gamma_i = \sum_{j=1}^i (-1)^j \bar{\alpha}_j \alpha_{i-j+1}$ . We note again that, in this case,  $\alpha_{n-2}$  appears only in  $\gamma_{n-2}$ , and we can write the sum as

$$\sum \cdots \sum_{\alpha_1, \alpha_{n-2}} \cdots q^2 \psi_0(\Delta d_0(-\bar{\alpha}_1 \alpha_{n-2} + \bar{\alpha}_{n-2} \alpha_1)).$$

We sum next over  $\alpha_1$  and  $\alpha_{n-2}$ , and we continue with this process to obtain at the end that (1) is  $(q^2)^{\frac{n}{2}} = q^n$ .



We assume this time  $n$  is odd. In this case, we have

$$\sum_{y \in A_n} \gamma(t, y) = \sum_{\substack{\alpha_i \\ i=0, \dots, n-1}} \psi_0(\Delta d_{n-1}\beta_0 + d_{n-2}\beta_1 + \dots + \Delta d_1\beta_{n-2} + d_0\beta_{n-1}). \quad (2)$$

As before,  $\alpha_{n-1}$  appears only in  $\beta_{n-1}$ . Summing first over  $\alpha_0$  and  $\alpha_{n-1}$ , to get

$$\begin{aligned} & \sum \dots \sum_{\alpha_0, \alpha_{n-1}} \dots \psi_0(\Delta d_0(\bar{\alpha}_0\alpha_{n-1} + \bar{\alpha}_{n-1}\alpha_0)) \\ &= \sum \dots \sum_{\alpha_0, \alpha_{n-1}} \dots \psi_0(\Delta d_0(\bar{\alpha}_0\alpha_{n-1} + \overline{\bar{\alpha}_0\alpha_{n-1}})). \end{aligned}$$

We have  $\psi_0(\Delta d_0(\bar{\alpha}_0\alpha_{n-1} + \overline{\bar{\alpha}_0\alpha_{n-1}})) = \psi_0(\Delta d_0 \operatorname{Tr}((\bar{\alpha}_0\alpha_{n-1})))$ , where  $\operatorname{Tr}$  stands for the field trace of the extension  $E \supset F$ .

Arguing as before, at the last step, i.e., after  $\frac{n-1}{2}$  steps we get

$$(q^2)^{\frac{n-1}{2}} \sum_{\alpha_{\frac{n-1}{2}}} \psi_0(\Delta \overline{\alpha_{\frac{n-1}{2}}} \alpha_{\frac{n-1}{2}}).$$

Using Lemma 3, we obtain (2) is equal to  $-q^n$ . ■

## 5 A first decomposition

We will get a first decomposition of  $\rho$ , constructed in Theorem 2. To this end, we lean on Theorem 7.6 in [7]. The unitary group  $U = U(\chi, \gamma, c)$  consisting of all  $A_n$ -linear automorphism  $\varphi$  of  $M$  such that  $\gamma(b, \varphi(x)) = \gamma(b, x)$ , for any  $b \in A_n$  and  $x \in M$ , allows us to obtain a decomposition of the Weil representation. In fact, the characters of  $U$  define the invariant subspaces of a decomposition of  $\rho$ . So, we will devote ourselves to obtain the structure of this group. Observing first that  $U$  is abelian, Theorem 7.6 in [7] reads as:

If  $\Lambda \in \hat{U}$ , let  $W_\Lambda$  be the vector subspace of  $W$  of the  $\Lambda$ -homogeneous functions, that is, the vector subspace of the functions  $f \in W$  such that  $f(ua) = \Lambda(u)f(a)$ , for  $a \in A_n$  and  $u \in U$ , then

**Theorem 3.** *The Weil representation  $(W, \rho)$  is the direct sum of all  $W_\Lambda$ , where  $\Lambda$  runs over all linear characters of  $U$ .*

We first prove:

**Proposition 6.** *We have*

1. *The group  $A_n^\times$  acts transitively on  $A_n^{\text{sym}} \cap A_n^\times$  by  $a \cdot t = ata^*$ .*
2. *The group of units  $A_n^\times$  acts transitively on  $A_n^{\text{asym}} \cap A_n^\times$  under the same action.*

**Proof.** 1. We consider the case when  $n$  is even. Since the ring  $A_n$  is commutative, we see that if  $t_1, t_2$  are in the orbit  $\operatorname{Orb}(1)$ , then  $t_1 t_2 \in \operatorname{Orb}(1)$ . We will first prove that every element of the form  $t = 1 + a_1 \Delta x + a_2 x^2 + \dots + a_{n-1} \Delta x^{n-1}$  ( $a_i \in k$ , for all  $i$ ) belongs to  $\operatorname{Orb}(1)$ .

In fact, one can check that there is an element  $b = 1 + b_1 \Delta x + b_2 x^2 + \dots + b_{n-1} \Delta x^{n-1} \in A_n$  ( $b_i \in k$ ), such that  $t = bb^*$ , given that the system

$$\begin{aligned} 1 &= 1, \\ 2b_1 &= a_1 \\ 2b_2 + b_1^2 \Delta^2 &= a_2, \end{aligned}$$



prove now that  $d_{n-2} = 0$ ) we take  $\lambda_0 = \lambda_1 = 1$  and  $\lambda_i = 0$  for  $i > 1$ . As before, we set  $y_i = 0$  for  $i > 0$  and  $y_0$  arbitrary, getting this time  $\psi_0(d_{n-2}\Delta\bar{y}_0y_0) = 1$ . So,  $d_{n-2} = 0$ .

Continuing with this process, we obtain  $d_i = 0$  for all  $i > 0$ . Finally, considering  $\lambda_0 = 1$  and  $\lambda_i = 0$  for  $i > 0$ , we are left with

$$\begin{aligned} & \psi_0(d_0\Delta(\bar{y}_0y_{n-1} - \bar{y}_1y_{n-2} + \cdots + (-1)^{n-1}\bar{y}_{n-1}y_0)) \\ &= \psi_0(\Delta(\bar{y}_0y_{n-1} - \bar{y}_1y_{n-2} + \cdots + (-1)^{n-1}\bar{y}_{n-1}y_0)), \end{aligned}$$

from which it follows  $d_0 = 1$ . ■

**Proposition 8.** *We have*

1.  $U \simeq N_{K/k}^{-1}(1) \times U_0$  where  $U_0 = \{b : b \in U, (b)_0 = 1\}$ ,  $(b)_0$  as defined in Proposition 5.
2.
  - If  $n$  is even, then the group  $U_0$  consists of all elements  $z$  of the form:

$$\begin{aligned} z = & 1 + \lambda_1x + (f_1(\lambda_1) + \lambda_2\Delta)x^2 + (\lambda_3 + f_2(\lambda_1, \lambda_2)\Delta)x^3 + \cdots \\ & + (\lambda_{n-1} + f_{n-2}(\lambda_1, \dots, \lambda_{n-2})\Delta)x^{n-1}, \end{aligned}$$

with  $\lambda_i \in k$ .

- If  $n$  is odd, then the group  $U_0$  consists of all elements  $z$  of the form:

$$\begin{aligned} z = & 1 + \lambda_1x + (f_1(\lambda_1) + \lambda_2\Delta)x^2 + (\lambda_3 + f_2(\lambda_1, \lambda_2)\Delta)x^3 + \cdots \\ & + (f_{n-2}(\lambda_1, \dots, \lambda_{n-2}) + \lambda_{n-1}\Delta)x^{n-1}, \end{aligned}$$

where  $\lambda_i \in k$ .

**Proof.** Part 1 follows directly from the definitions.

We prove part 2. If  $b = r + c\Delta$ , we will write  $r = \text{Real}(b) \in k$  and  $c = \text{Im}(b) \in k$ . Expanding

$$(1 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1})(1 - \bar{b}_1x + \bar{b}_2x^2 - \cdots \pm \bar{b}_{n-1}x^{n-1}) = 1,$$

we see that  $b_1 \in k$ . We set  $\lambda_1 = b_1$ .

Similarly, we get  $b_2 + \bar{b}_2 = b_1\bar{b}_1$  so  $\text{Real}(b_2) = \frac{\lambda_1^2}{2}$ . We see  $\text{Im}(b_2)$  is an element of  $k$ , independent of  $\lambda_1$ . Setting  $\lambda_2 = \text{Im}(b_2)$ , we can write  $b_2 = f_1(\lambda_1) + \lambda_2\Delta$  with  $f_1(\lambda_1)$  a function of  $\lambda_1$  which is independent from  $\lambda_2$ .

In the next step, we obtain  $b_3 - \bar{b}_3 = b_2\bar{b}_1 - b_1\bar{b}_2$ , from which  $\text{Im}(b_3) = 2\lambda_1\lambda_2$ . We observe  $\text{Real}(b_3)$  is an element of  $k$  independent from  $\lambda_1$  and  $\lambda_2$ . We write this time  $b_3 = \lambda_3 + f_2(\lambda_1, \lambda_2)\Delta$ , for a function  $f_2(\lambda_1, \lambda_2)$  independent from  $\lambda_3$ .

In general, when  $i$  is even,  $b_i + \bar{b}_i$  is a  $k$ -valued function  $f_i(\lambda_1, \dots, \lambda_{i-1})$ , and  $\text{Im}(b_i) = f_{i-1}(\lambda_1, \dots, \lambda_{i-1})$  is a new variable, getting  $b_i = \lambda_i$ . When  $i$  is odd,  $b_i - \bar{b}_i$  determines a  $k$ -valued function, we set this time  $\text{Im}(b_i) = f_{i-1}(\lambda_1, \dots, \lambda_{i-1})$  and  $\text{Real}(b_i) = f_{i-1}(\lambda_1, \dots, \lambda_{i-1})$  for a new variable independent from  $\lambda_1, \dots, \lambda_{i-1}$ . We can write  $b_i = \lambda_i + f_{i-1}(\lambda_1, \dots, \lambda_{i-1})\Delta$ . The result now follows. ■

**Remark 1.** We observe that in fact the functions  $f_i$  have the property

$$f_i(0, 0, \dots, 0) = 0.$$

The field  $k = \mathbb{F}_q$  (where  $q = p^t$ ,  $p$  an odd prime number) is a  $t$ -dimensional vector space over  $\mathbb{F}_p$ . We set  $e_1, \dots, e_t$  for a basis of  $k$  over  $\mathbb{F}_p$ .

We describe the elements of  $U_0$  as in Proposition 8, and we define:

**Definition 1.** Let  $i$  be relatively prime to  $p$ , and  $l = 1, \dots, t$ .  $H_{i,l}$  denotes the cyclic subgroup of  $U_0$  of order  $d = d_{i,l}$  ( $d$  is a power of  $p$ ), generated by  $z = 1 + e_l x^i + \alpha_2 x^{2i} + \dots + \alpha_{\text{ord}(z)} x^{\text{ord}(z)i}$  for  $i$  odd, and generated by  $z = 1 + e_l \Delta x^i + \alpha_2 \Delta^2 x^{2i} + \dots + \alpha_{\text{ord}(z)} \Delta^{\text{ord}(z)} x^{\text{ord}(z)i}$  for  $i$  even, where  $\text{ord}(z)$  is the integer such that  $\text{ord}(z)i < n$ , but  $(\text{ord}(z) + 1)i \geq n$ , and  $\alpha_j$  are certain elements in  $k$ .

We determine now the elements  $\alpha_i$  for  $i$  odd (the other case being similar). The condition

$$(1 + e_l x^i + \alpha_2 x^{2i} + \dots)(1 + e_l x^i + \alpha_2 x^{2i} + \dots)^* = 1$$

leads to the system (with  $\alpha_1 = e_l$ )

$$\begin{aligned} 1 &= 1, \\ 2\alpha_2 - \alpha_1^2 &= 0, \\ 0 &= 0, \\ 2\alpha_4 - 2\alpha_3 + \alpha_2^2 &= 0, \\ &\dots\dots\dots \\ \sum_{s=1}^j (-1)^{s-j} \alpha_s \alpha_{s-j} &= 0, \end{aligned}$$

which has the solution  $\alpha_2 = \frac{1}{2}\alpha_1^2$ ,  $\alpha_j = 0$  for  $j$  odd and for  $j$  even greater than 2. So,  $\alpha_j$  is computed inductively by  $2\alpha_j + \sum_t \alpha_{2t} \alpha_{j-2t} = 0$ .

**Proposition 9.** The intersection of  $\prod_{v,u} H_{v,u}$  with the subgroup  $H_{i_0, l_0}$  is 1 [ $1 \leq v \leq n-1$ ,  $v$  relatively prime to  $p$ ,  $v > i_0$  if  $u = l_0$ , and  $v \geq i_0$  if  $u \neq l_0$ ,  $u = 1, \dots, t$ ].

**Proof.** Let  $H_{i_0, l_0}$  with  $i_0$  relatively prime to  $p$ . We prove the case when  $i_0$  is odd (the case when  $i_0$  is even is similar).

Let  $z \in H_{i_0, l_0}$ . Then  $z = (1 + e_{l_0} x^{i_0} + \alpha_2 x^{2i_0} + \dots + \alpha_{\text{ord}(z)} x^{\text{ord}(z)i_0})^j$ , where  $1 \leq j \leq d$  ( $d$  is a power of  $p$ ) is such that  $i_0 d > n$ . We have  $j = p^a s$ , where  $a > 0$ , g.c.d.  $(s, p) = 1$  if  $p$  divides  $j$ , and  $a = 0$  if  $j = s$  is relatively prime to  $p$ . Then, we can write  $z = 1 + s_0 e_{l_0} x^{p^a i_0} +$  terms of higher degree ( $s \equiv s_0 \pmod{p}$ ,  $1 \leq s_0 < p$ ).

If  $z$  is also an element of the product above, then

$$z = (1 + \beta_1 e_{l_1} x^{p^{c_1} i_{l_1}} + \dots)(1 + \beta_2 e_{l_2} x^{p^{c_2} i_{l_2}} + \dots) \dots,$$

( $1 \leq \beta_r < p$ ,  $r = 1, \dots, m$ ) for some  $m$ , and  $i_r > i_0$  if  $l_r = l_0$ ,  $i_r \geq i_0$  if  $l_r \neq l_0$ . So, the lower degree term of  $z$ , as an element of the product must be of the form  $(\beta_1 e_{l_1} + \beta_2 e_{l_2} + \dots + \beta_m e_{l_m}) x^{p^a i_0}$  with  $p^{c_1} i_{l_1} = \dots = p^{c_m} i_{l_m} = p^a i_0$ . But then  $\beta_1 e_{l_1} + \beta_2 e_{l_2} + \dots + \beta_m e_{l_m} = s_0 e_{l_0}$ .

We have two possible cases according to whether  $l_0$  appears in the factors for  $z$  (as an element of a product of  $H$ 's) or not. In the first case, since the  $e$ 's are linearly independent, we must have (reordering if necessary)  $\beta_1 = s_0$ ,  $\beta_2 = \dots = \beta_m = 0$  and  $l_1 = l_0$ . The equality  $p^{c_1} i_1 = p^a i_0$  is contradictory because  $i_1 > i_0$ , and  $i_0$  and  $i_1$  are relatively prime to  $p$ . In the second case, we would have linear dependency between the  $e$ 's.

From here, the result follows. ■

The next lemmas are direct result of the definition of the integer part function.

**Lemma 5.** Let  $a, b$  be positive integers such that  $a < b$ , then the number of multiples of  $p$  in the interval  $[a, b]$  is  $\lfloor \frac{b}{p} \rfloor - \lfloor \frac{a}{p} \rfloor$ .

**Lemma 6.** *Let  $n, p$  be positive integers with  $p$  a prime element. Then*

$$\left\lfloor \frac{\left\lfloor \frac{n}{p^j} \right\rfloor}{p} \right\rfloor = \left\lfloor \frac{n}{p^{j+1}} \right\rfloor.$$

We assume  $n$  is relatively prime to  $p$ . Consider the maximal integer  $r$  such that  $n = p^{r-1}m + R$ , with  $0 < R < p^{r-1}$ .

**Proposition 10.** *The cardinality of the subgroup  $H_{i,u}$ , for  $u = 1, \dots, t$ , where  $i$  belongs to the interval  $]\lfloor \frac{n}{p^j} \rfloor, \lfloor \frac{n}{p^{j-1}} \rfloor]$  and  $i$  is relatively prime to  $p$ , is  $p^j$ .*

**Proof.** We observe that to find the order of  $H_{i,u}$  we need to find the integer  $\alpha_i$  such that  $ip^{\alpha_i-1} < n < ip^{\alpha_i}$ , from which  $H_{i,u}$  will have order  $p^{\alpha_i}$ . Let us write  $n = a_{r-1}p^{r-1} + a_{r-2}p^{r-2} + \dots + a_1p + a_0$  with  $0 \leq a_i < p$  and  $a_{r-1} \neq 0$ . Then

$$\begin{aligned} \left\lfloor \frac{n}{p^j} \right\rfloor &= a_{r-1}p^{r-1-j} + a_{r-2}p^{r-2-j} + \dots + a_j, \\ \left\lfloor \frac{n}{p^{j-1}} \right\rfloor &= a_{r-1}p^{r-j} + a_{r-2}p^{r-1-j} + \dots + a_jp + a_{j-1}. \end{aligned}$$

The conditions on  $i$  and  $n$  say

$$a_{r-1}p^{r-1-j} + a_{r-2}p^{r-2-j} + \dots + a_j < i \leq a_{r-1}p^{r-j} + a_{r-2}p^{r-1-j} + \dots + a_jp + a_{j-1},$$

from which we have the inequalities

$$p^{j-1}i \leq a_{r-1}p^{r-1} + a_{r-2}p^{r-2} + \dots + a_jp^j + a_{j-1}p^{j-1} \leq n,$$

but since  $n$  is relatively prime to  $p$ , we have  $p^{j-1}i < n$ , and

$$a_{r-1}p^{r-1} + a_{r-2}p^{r-2} + \dots + a_jp^j < p^ji.$$

Using the base  $p$  expansion of  $n$ , we get from this last inequality that  $n < p^ji$ . Taking  $\alpha_i = j$  the result follows.  $\blacksquare$

**Corollary 1.** *The group  $U_0$  is the direct product of all subgroups  $H_{i,u}$  as above, where  $u$  varies from 1 to  $t$ , and  $i$  is less or equal to  $n-1$  and relatively prime to  $p$ .*

**Proof.** According to Propositions 9, 10 and Lemma 4 it is enough to show that the order of the direct product of all subgroups  $H_{i,u}$  is  $q^{n-1}$ .

Let us fix an element  $u$  ( $u = 1, \dots, t$ ). First, notice that there are no multiples of  $p$  in the interval  $[1, \lfloor \frac{n}{p^{r-1}} \rfloor]$ . In general, the number of multiples of  $p$  in  $[\lfloor \frac{n}{p^{r-j}} \rfloor + 1, \lfloor \frac{n}{p^{r-j-1}} \rfloor]$  is

$$\left\lfloor \frac{\lfloor \frac{n}{p^{r-j-1}} \rfloor}{p} \right\rfloor - \left\lfloor \frac{\lfloor \frac{n}{p^{r-j}} \rfloor}{p} \right\rfloor = \left\lfloor \frac{n}{p^{r-j}} \right\rfloor - \left\lfloor \frac{n}{p^{r-j+1}} \right\rfloor.$$

For each integer  $j \geq 0$ , we denote by  $\gamma_j$  the number of integers relatively prime to  $p$  in the interval  $[\lfloor \frac{n}{p^{r-j}} \rfloor + 1, \lfloor \frac{n}{p^{r-j-1}} \rfloor]$ . We observe that

$$\gamma_j = \left( \left\lfloor \frac{n}{p^{r-j-1}} \right\rfloor - \left\lfloor \frac{n}{p^{r-j}} \right\rfloor \right) - \left( \left\lfloor \frac{n}{p^{r-j}} \right\rfloor - \left\lfloor \frac{n}{p^{r-j+1}} \right\rfloor \right).$$

Then, the order of the subgroup  $H_{i,u}$  with  $i$  in the interval  $[\lfloor \frac{n}{p^{r-j}} \rfloor + 1, \lfloor \frac{n}{p^{r-j-1}} \rfloor]$  is  $p^{r-j}$ . Hence the product of the orders of all  $H_{i,u}$ , for a fixed  $u$ , is  $p^{\sum_{j=0}^{r-1} \gamma_j(r-j)}$ . Now, we verify that

$$\begin{aligned} \sum_{j=0}^{r-1} \gamma_j(r-j) &= r \left\lfloor \frac{n}{p^{r-1}} \right\rfloor + (r-1) \left[ \left( \left\lfloor \frac{n}{p^{r-2}} \right\rfloor - \left\lfloor \frac{n}{p^{r-1}} \right\rfloor \right) - \left\lfloor \frac{n}{p^{r-1}} \right\rfloor \right] \\ &\quad + (r-2) \left[ \left( \left\lfloor \frac{n}{p^{r-3}} \right\rfloor - \left\lfloor \frac{n}{p^{r-2}} \right\rfloor \right) - \left( \left\lfloor \frac{n}{p^{r-2}} \right\rfloor - \left\lfloor \frac{n}{p^{r-1}} \right\rfloor \right) \right] \\ &\quad + (r-3) \left[ \left( \left\lfloor \frac{n}{p^{r-4}} \right\rfloor - \left\lfloor \frac{n}{p^{r-3}} \right\rfloor \right) - \left( \left\lfloor \frac{n}{p^{r-3}} \right\rfloor - \left\lfloor \frac{n}{p^{r-2}} \right\rfloor \right) \right] \\ &\quad + (r-4) \left[ \left( \left\lfloor \frac{n}{p^{r-5}} \right\rfloor - \left\lfloor \frac{n}{p^{r-4}} \right\rfloor \right) - \left( \left\lfloor \frac{n}{p^{r-4}} \right\rfloor - \left\lfloor \frac{n}{p^{r-3}} \right\rfloor \right) \right] + \dots \\ &\quad + 3 \left[ \left( \left\lfloor \frac{n}{p^2} \right\rfloor - \left\lfloor \frac{n}{p^3} \right\rfloor \right) - \left( \left\lfloor \frac{n}{p^3} \right\rfloor - \left\lfloor \frac{n}{p^4} \right\rfloor \right) \right] \\ &\quad + 2 \left[ \left( \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{n}{p^2} \right\rfloor \right) - \left( \left\lfloor \frac{n}{p^2} \right\rfloor - \left\lfloor \frac{n}{p^3} \right\rfloor \right) \right] \\ &\quad + \left[ \left( (n-1) - \left\lfloor \frac{n}{p} \right\rfloor \right) - \left( \left\lfloor \frac{n}{p} \right\rfloor - \left\lfloor \frac{n}{p^2} \right\rfloor \right) \right] = n-1. \end{aligned}$$

From this, it is clear that the product of all  $H_{i,u}$  (running also over  $u$ ) is  $(p^{n-1})^t = q^{n-1}$ . From here the result follows.  $\blacksquare$

We have found a decomposition of  $U$  as a product of cyclic subgroups:

**Theorem 4.** *The group  $U$  is the direct product of a cyclic subgroup of order  $q+1$  and the cyclic subgroups determined by the groups  $H_{i,l}$  of Corollary 1.*

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